11 - Introduction to Jump Processes

Stochastic Calculus for Finance

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11.1 Introduction
Jump-diffusion processes

- The “Diffusion” means these processes can have a **Brownian motion.**
    - More generally, an **integral with respect to Brownian motion.**
- The paths of these processes may have jumps.
- We consider in this chapter the special case when there are only finitely many jumps in each finite time interval.
- The number of jumps depends on the threshold.
    - As the threshold approaches zero, it becomes arbitrarily large.

11.1 Introduction
Section 11.2 - 11.4

- The fundamental pure jump process is the ***Poisson process (Section 11.2)***
  - All jumps of a Poisson process are of size one.
- ***Compound Poisson process (Section 11.3)*** is like a Poisson process, except that the jumps are random size.
- Define a ***jump process*** to be the sum of a nonrandom initial condition, an Itô integral with respect to a Brownian motion $dW(t)$, a Rieman integral with respect to dt, and a ***pure jump*** process. ***(Section 11.4)***

11.1 Introduction

Section 11.5 - 11.7

- The stochastic calculus for jump process ***(Section 11.5)***
  - The key result is the extension of the Itô-Doeblin formula to cover these processes.
- Changing the measures for Poisson processes and compound Poisson processes. ***(Section 11.6)***
  - How to simultaneously change the measure for a Brownian motion and a compound Poisson process.
  - The effect of this change is to adjust the drift of the Brownian motion and to adjust the *intensity* **(average rate of jump arrival)** and the distribution of the jump sizes for the compound Poisson process
- Apply this theory to the problem of **pricing and partially hedging a European call** in a **jump-diffusion model *(Section 11.7)***

11.2 Poisson Process

11.2.1 Exponential Random Variables

- Let τ be a random variable with density

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (11.2.1)$$

- Where λ is a positive constant. We say that τ has the *exponential distribution* or simply that τ is an *exponential random variable*.
- The expected value of τ can be computed by an integration by parts:

$$\mathbb{E}_\tau = \int_0^\infty t f(t) dt = \lambda \int_0^\infty t e^{-\lambda t} dt$$

$$= \lambda(-t\frac{1}{\lambda}e^{-\lambda t}|_{t=0}^{t=\infty} - \int_0^\infty -\frac{1}{\lambda}e^{-\lambda t}dt)$$

$$= -te^{-\lambda t}|_{t=0}^{t=\infty} + \int_0^\infty e^{-\lambda t}dt$$

$$= 0 - \frac{1}{\lambda}e^{-\lambda t}|_{t=0}^{t=\infty}$$

$$= 0 - (0 - \frac{1}{\lambda}e^0)$$

$$= \frac{1}{\lambda}$$

$$\int u dv = uv - \int v du$$

$u = t,$

$du = dt,$

$v = -\frac{1}{\lambda}e^{-\lambda t},$

$dv = e^{-\lambda t}dt$

11.2 Poisson Process

11.2.1 Exponential Random Variables

- For the cumulative distribution function, we have

$$F(t) = \mathbb{P}\{\tau \le t\} = \int_0^t \lambda e^{-\lambda u} du = -e^{-\lambda u}|_{u=0}^{u=t} = 1 - e^{-\lambda t}, t \ge 0,$$

and hence

$\mathbb{P}\{\tau > t\} = 1 - \mathbb{P}\{\tau \le t\} = e^{-\lambda t}, t \ge 0$ (11.2.2)

11.2 Poisson Process

11.2.1 Exponential Random Variables

- Suppose we are waiting for an event, such as default of a bond, and we know that the distribution of the time of this event is exponential with mean $\frac{1}{\lambda}$
- Suppose we have already waited **s** time units, and we are interested in the probability that we will have to wait an additional **t** time units (conditioned on knowing that the event has not occurred during the time interval [0, s]).

11.2 Poisson Process

11.2.1 Exponential Random Variables

- This probability is

$$\mathbb{P}\{\tau > t+s \mid \tau > s\} = \frac{\mathbb{P}\{\tau > t+s \text{ and } \tau > s\}}{\mathbb{P}\{\tau > s\}}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\mathbb{P}\{\tau > t+s\}}{\mathbb{P}\{\tau > s\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} \quad (11.2.3)$$

$$\mathbb{P}\{\tau > t\} = 1 - \mathbb{P}\{\tau \leq t\} = e^{-\lambda t}, t \geq 0 \quad (11.2.2)$$

- Starting from **time s** and **time 0**, the probabilities of both are the **same**!
- The fact that we have already waited s time units does not change the distribution of the remaining time.
    - This property for the exponential distribution is called *memorylessness*

11.2 Poisson Process

11.2.2 Construction of a Poisson Process

- Begin with a sequence $\tau_1, \tau_2, ...$ of independent exponential variables, all with the same mean $\frac{1}{\lambda}$
- Build a model in which an event, which we call a **“jump”**, **occurs from time to time.**
- The τ_k random variables are called the ***interarrival times***. The ***arrival times*** are

$$S_n = \sum_{k=1}^{n} \tau_k \text{ (11.2.4)}$$

- S_n is the time of the *n*th jump

11.2 Poisson Process

11.2.2 Construction of a Poisson Process

- The Poisson process N(t) counts the number of jumps that occur at or before time t

$$N(t) = \begin{cases} 0 \text{ if } 0 \le t < S_1, \\ 1 \text{ if } S_1 \le t < S_2, \\ \vdots \\ n \text{ if } S_n \le t < S_{n+1}, \\ \vdots \end{cases}$$

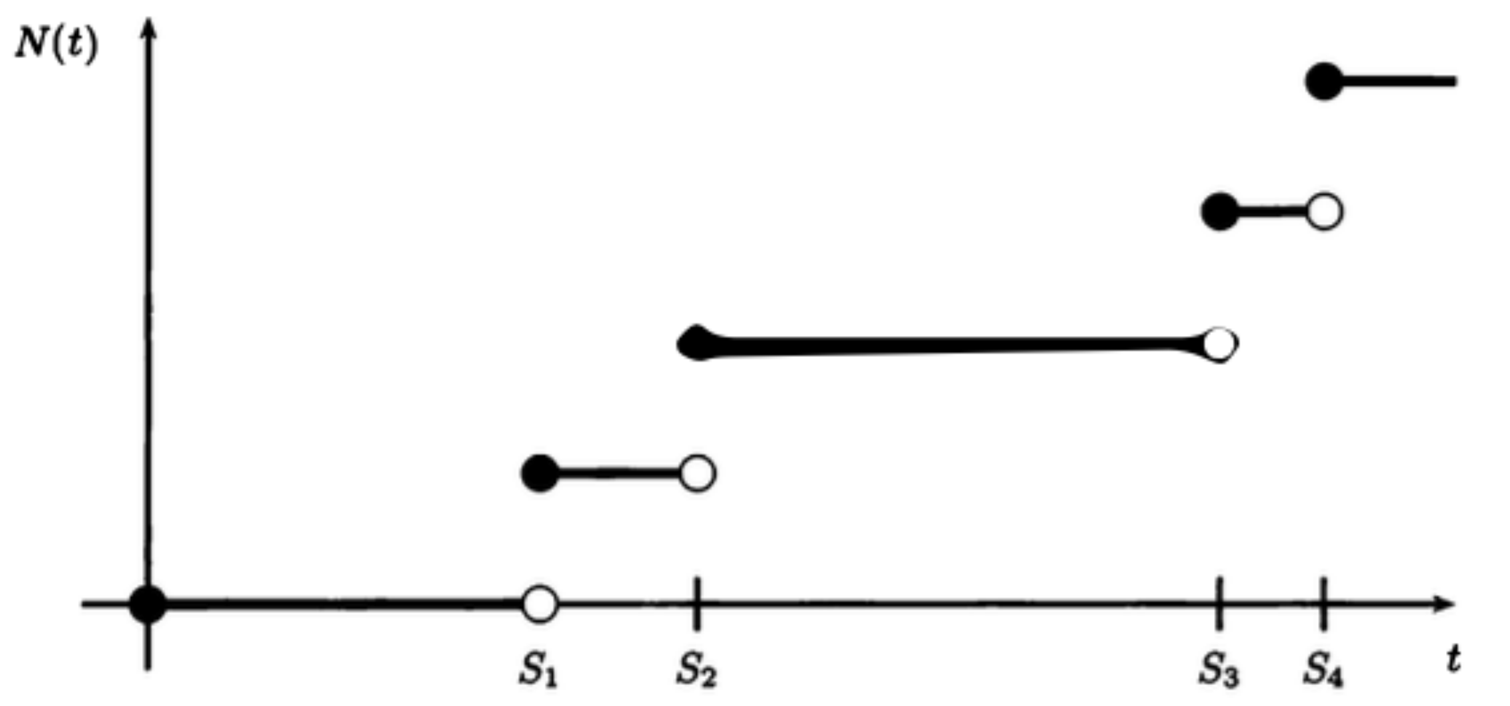

Fig. 11.2.1. One path of a Poisson process.

- Note that at the jump times N(t) is defined so that it is *right-continuous* (i.e., $N(t) = lim_{s \downarrow t} N(s)$)

- We denote by $F(t)$ the σ-algebra of information acquired by observing $N(s)$ for $0 \le s \le t$

11.2 Poisson Process

11.2.2 Construction of a Poisson Process

- Because the expected time between jumps is $\frac{1}{\lambda}$, the jumps are arriving at an average rate of λ per unit time.
- We say the Poisson process $N(t)$ has *intensity* λ

Thanks for listening